ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

J. Math. Anal. Appl. 337 (2008) 919–927

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

# The local inverse mapping theorem on Banach orbifolds

Guangcun Lu [*,1], MingYan Wang

*Department of Mathematics, Beijing Normal University, Beijing 100875, PR China*

Received 27 September 2006

Available online 14 February 2007

Submitted by M. Milman

**Abstract**

In this note, the local inverse mapping theorem is extended to a class of Banach orbifolds.



*Keywords:* Banach orbifold; Inverse mapping theorem

## 1. Introduction and Banach orbifolds

Recently, Carlotti [1] extended the local inverse theorem to V-manifolds of finite dimensions. V-manifolds or orbifolds were first introduced by Satake [2], and arise naturally in many ways. Unlike direct extension of the local inverse theorem from finitely dimensional manifolds to Banach manifolds, when trying to extend Carlotti's result to Banach orbifolds, we find the extension can only be done on a class of Banach orbifolds. Theorem 2.1 is our main result.

Recall that a ($C^\infty$-) *Banach orbifold chart* for an open subset $U$ of a Hausdorff topological space $X$ is a triple $(\widetilde{U}, G_U, \varphi_U)$ consisting of

(i) a connected open subset $\widetilde{U}$ of a Banach manifold;
(ii) a finite group $G_U$ of ($C^\infty$-) automorphisms of $\widetilde{U}$ which effectively acts on $\widetilde{U}$;

[*] Corresponding author.
*E-mail addresses:* gclu@bnu.edu.cn (G. Lu), wangmingyan@mail.bnu.edu.cn (M. Wang).

[1] Partially supported by the NNSF 10671017 of China and the Program for New Century Excellent Talents of the Education Ministry of China.

0022-247X/$ – see front matter 
doi:10.1016/j.jmaa.2007.02.007

(iii) a continuous surjective map $\varphi_U : \widetilde{U} \to U$ such that $\varphi_U \circ g = \varphi_U$ for any $g \in G_U$, and that the induced map $\widetilde{U}/G_U \to U$ is a homeomorphism.

One often calls $\widetilde{U}$ as a local cover, $G_U$ as a local group, $U$ as the support of the chart, $\varphi_U$ as a local covering map. Note that $\varphi_U : \widetilde{U} \to U$ is both open and closed, in particular it is proper by the proof of Proposition 1.7 on p. 102 of [3].

For any $x \in U$ and $\tilde{x} \in (\varphi_U)^{-1}(x)$, let $G_U(\tilde{x})$ denote the isotropy group of $G_U$ at $\tilde{x}$, i.e., $G_U(\tilde{x}) = \{g \in G_U \mid g(\tilde{x}) = \tilde{x}\}$. $\tilde{x}$ is called a *singular point* if $G_U(\tilde{x}) \neq \{I\}$, otherwise it is called a *regular point*. Denote $\widetilde{U}^{\text{sing}}$ (respectively $\widetilde{U}^{\circ}$) by the set of all singular (respectively regular) points of $\widetilde{U}$, i.e., $\widetilde{U}^{\text{sing}} = \{\tilde{x} \in \widetilde{U} \mid G_U(\tilde{x}) \neq \{I\}\}$ (respectively $\widetilde{U}^{\circ} = \{\tilde{x} \in \widetilde{U} \mid G_U(\tilde{x}) = \{I\}\}$). We also denote $G_U\tilde{x}$ by the orbit of $\tilde{x}$, i.e., $G_U\tilde{x} = \{g(\tilde{x}) \mid g \in G_U\}$.

**Lemma 1.1.** *Let $(\widetilde{U}, G_U, \varphi_U)$ be a Banach orbifold. Then for any $x \in U$ and $\tilde{x} \in (\varphi_U)^{-1}(x)$, there exists a connected open neighborhood $\widetilde{O}(\tilde{x}) \subset \widetilde{U}$ of $\tilde{x}$ such that*

(i) *$\widetilde{O}(\tilde{x})$ is $G_U(\tilde{x})$-invariant;*
(ii) *$h(\widetilde{O}(\tilde{x})) \cap \widetilde{O}(\tilde{x}) = \emptyset$ for any $h \in G_U \setminus G_U(\tilde{x})$.*

**Proof.** If $G_U(\tilde{x}) = G_U$, we can take $\widetilde{O}(\tilde{x}) = \widetilde{U}$. So we may assume $G_U(\tilde{x}) \neq G_U$. For any $h \in G_U \setminus G_U(\tilde{x})$, $h(\tilde{x}) \neq \tilde{x}$, we can always find an open neighborhood $V_h$ of $\tilde{x}$ such that $h(V_h) \cap V_h = \emptyset$. Then $V_1 := \bigcap_{h \in G_U \setminus G_U(\tilde{x})} V_h$ is also an open neighborhood of $\tilde{x}$ and satisfies $h(V_1) \cap V_1 = \emptyset$ for any $h \in G_U \setminus G_U(\tilde{x})$. Since $g(\tilde{x}) = \tilde{x}$ for any $g \in G_U(\tilde{x})$, there must exist an open neighborhood $V_g$ of $\tilde{x}$ such that $g(V_g) \subset V_1$. Set $V_2 := \bigcap_{g \in G_U(\tilde{x})} V_g$, we have $g(V_2) \subset V_1$ for any $g \in G_U(\tilde{x})$. Take a connected open neighborhood $V$ of $\tilde{x}$ such that $V \subset V_1 \cap V_2$. Then $\widetilde{O}(\tilde{x}) := \bigcup_{g \in G_U(\tilde{x})} g(V)$ is also a connected open neighborhood of $\tilde{x}$ and satisfies (i) obviously. To see that it also satisfies (ii), note that for any $h \in G_U \setminus G_U(\tilde{x})$,

$$h\big(\widetilde{O}(\tilde{x})\big) \cap \widetilde{O}(\tilde{x}) = \left(\bigcup_{g \in G_U(\tilde{x})} hg(V)\right) \cap \left(\bigcup_{g' \in G_U(\tilde{x})} g'(V)\right)$$
$$= \bigcup_{g \in G_U(\tilde{x})} \bigcup_{g' \in G_U(\tilde{x})} \big(hg(V) \cap g'(V)\big).$$

Since $hg \in G_U \setminus G_U(\tilde{x})$, $V \subset V_1$ implies $hg(V) \cap V_1 = \emptyset$, and $V \subset V_2$ implies $g'(V) \subset V_1$. Hence we get $hg(V) \cap g'(V) = \emptyset$, and thus $h(\widetilde{O}(\tilde{x})) \cap \widetilde{O}(\tilde{x}) = \emptyset$. $\square$

If $\widetilde{U}$ is of finite dimension, then $\widetilde{U}^{\circ}$ is a dense open subset in $\widetilde{U}$ by a theorem by Newman [4, Theorem 1]. However, for infinitely dimensional $\widetilde{U}$, we can not confirm such a conclusion. In order to find an appropriate extension of it, let us firstly analysis the local structure of $\widetilde{U}^{\text{sing}}$. For any $\tilde{x} \in \widetilde{U}$, if $G_U(\tilde{x}) \neq \{I\}$, i.e., $\tilde{x} \in \widetilde{U}^{\text{sing}}$, it follows from the local linearizion theorem in [5] that there exists a $G_U(\tilde{x})$-equivariant diffeomorphism $F : N(0_{\tilde{x}}) \to \mathcal{N}(\tilde{x})$, i.e., $F(dg(\tilde{x})\xi) = g(F(\xi))$ $\forall g \in G_U(\tilde{x})$ and $\forall \xi \in N(0_{\tilde{x}})$, where $N(0_{\tilde{x}})$ is a neighborhood of the origin in $T_{\tilde{x}}\widetilde{U}$, and $\mathcal{N}(\tilde{x})$ is a neighborhood of $\tilde{x}$ in $\widetilde{U}$. By Lemma 1.1, this $\mathcal{N}(\tilde{x})$ can be chosen to be connected and so small that

(i) $\mathcal{N}(\tilde{x})$ is $G_U(\tilde{x})$-invariant;
(ii) $h(\mathcal{N}(\tilde{x})) \cap \mathcal{N}(\tilde{x}) = \emptyset$ for any $h \in G_U \setminus G_U(\tilde{x})$.

Let $\mathcal{V}(dg(\tilde{x})) := \mathrm{Ker}(id_{T_{\tilde{x}}\widetilde{U}} - dg(\tilde{x}))$. We claim

$$\widetilde{U}^{\mathrm{sing}} \cap \mathcal{N}(\tilde{x}) = \mathcal{N}(\tilde{x})^{\mathrm{sing}} = \bigcup_{g \in G_U(\tilde{x}) \setminus \{I\}} F\big(N(0_{\tilde{x}}) \cap \mathcal{V}\big(dg(\tilde{x})\big)\big). \tag{1}$$

In fact, for any $\tilde{y}$ in the left side, there exists $g_0 \in G_U$, $g_0 \neq I$ such that $g_0(\tilde{y}) = \tilde{y}$. Note that $\tilde{y} \in \mathcal{N}(\tilde{x})$, it follows from (ii) that $g_0 \in G_U(\tilde{x})$. Since $F : N(0_{\tilde{x}}) \to \mathcal{N}(\tilde{x})$ is a diffeomorphism, there exists $\xi \in N(0_{\tilde{x}})$ such that $\tilde{y} = F(\xi)$. Now $F(dg_0(\tilde{x})\xi) = g_0(F(\xi)) = g_0(\tilde{y}) = \tilde{y} = F(\xi)$, we have $\xi \in N(0_{\tilde{x}}) \cap \mathcal{V}(dg_0(\tilde{x}))$, which implies $\tilde{y}$ is in the right side. Hence the left side is contained in the right side. Conversely, it is easily checked that the right side is contained in the left side. (1) is proved.

So for any $g \in G_U(\tilde{x}) \setminus \{I\}$, the submanifold $F(N(0_{\tilde{x}}) \cap \mathcal{V}(dg(\tilde{x}))) \subset \mathcal{N}(\tilde{x})$ is exactly the set of the fixed points of $g$ in $\mathcal{N}(\tilde{x})$, i.e., $\mathcal{N}(\tilde{x})^g$, and $\mathcal{N}(\tilde{x})^{\mathrm{sing}} = \bigcup_{g \in G_U(\tilde{x}) \setminus \{I\}} \mathcal{N}(\tilde{x})^g$. We define

$$\mathrm{codim}\, \mathcal{N}(\tilde{x})^{\mathrm{sing}} = \min\big\{\mathrm{codim}\, \mathcal{N}(\tilde{x})^g \;\big|\; g \in G_U(\tilde{x}) \setminus \{I\}\big\},$$

and call it *codimension* of $\widetilde{U}^{\mathrm{sing}}$ near $\tilde{x}$. When $\mathrm{codim}\, \mathcal{N}(\tilde{x})^{\mathrm{sing}} \geqslant 1$, it is easily seen that the set of regular points of $\mathcal{N}(\tilde{x})$, i.e., $\mathcal{N}(\tilde{x})^\circ$, is a dense open subset in $\mathcal{N}(\tilde{x})$. For integer $k \geqslant 1$, we say the Banach orbifold chart $(\widetilde{U}, G_U, \varphi_U)$ to be *k-regular* if the codimension of $\widetilde{U}^{\mathrm{sing}}$ near any $\tilde{x} \in \widetilde{U}^{\mathrm{sing}}$ is not less than $k$. It means that the fixed point set $\widetilde{U}^g$ of any $g \in G_U \setminus \{I\}$ has codimension not less than $k$ near any $\tilde{x} \in \widetilde{U}^g$. So for any $k$-regular Banach orbifold chart $(\widetilde{U}, G_U, \varphi_U)$, $\widetilde{U}^\circ$ is a dense open subset in $\widetilde{U}$.

**Lemma 1.2.** *Let $(\widetilde{U}, G_U, \varphi_U)$ be k-regular Banach orbifold chart, $x \in U$, $\tilde{x} \in (\varphi_U)^{-1}(x)$, and $\widetilde{O}(\tilde{x}) \subset \widetilde{U}$ be a connected open neighborhood of $\tilde{x}$ as in Lemma* 1.1. *Then $(\widetilde{O}(\tilde{x}), G_U(\tilde{x}), \varphi_U^x = \varphi_U|_{\widetilde{O}(\tilde{x})})$ is also a k-regular Banach orbifold chart, called an* induced chart *of $(\widetilde{U}, G_U, \varphi_U)$ at $\tilde{x}$ (or $x$).*

**Proof.** Since $U^\circ$ is a dense open subset in $U$, we may assume $x \in U^{\mathrm{sing}}$. Firstly, it is clear that $O(x) = \varphi_U(\widetilde{O}(\tilde{x}))$ is an open subset in $U$ (and so in $X$) since $\varphi_U$ is an open map. Next, for any $g \in G_U(\tilde{x})$ and $\tilde{y} \in \widetilde{O}(\tilde{x})$, we have $\tilde{x}_{\tilde{y}} := g^{-1}(\tilde{y}) \in \widetilde{O}(\tilde{x})$ and $g(\tilde{x}_{\tilde{y}}) = \tilde{y}$ because $g^{-1} \in G_U(\tilde{x})$. It implies $g|_{\widetilde{O}(\tilde{x})} : \widetilde{O}(\tilde{x}) \to \widetilde{O}(\tilde{x})$ is a surjective map and thus a homeomorphism. Obviously the codimension of $\widetilde{O}(\tilde{x})^{\mathrm{sing}}$ near $\tilde{y}$ is the same as that of $\widetilde{U}^{\mathrm{sing}}$ near $\tilde{y}$. Finally, we show that $\varphi_U^x : \widetilde{O}(\tilde{x}) \to O(x)$ induces the following homeomorphism

$$\overline{\varphi_U^x} : \widetilde{O}(\tilde{x})/G_U(\tilde{x}) \to O(x), \qquad [\tilde{y}]_{G_U(\tilde{x})} \mapsto \varphi_U^x(\tilde{y}) = \varphi_U(\tilde{y})$$

where $\tilde{y} \in \widetilde{O}(\tilde{x})$, $[\tilde{y}]_{G_U(\tilde{x})} := G_U(\tilde{x})\tilde{y}$. In fact, it suffices to prove that $\overline{\varphi_U^x}$ is injective and open. Assume $\overline{\varphi_U^x}([\tilde{y}_1]_{G_U(\tilde{x})}) = \overline{\varphi_U^x}([\tilde{y}_2]_{G_U(\tilde{x})})$ for $\tilde{y}_1, \tilde{y}_2 \in \widetilde{O}(\tilde{x})$. Then $\varphi_U^x(\tilde{y}_1) = \varphi_U(\tilde{y}_1) = \varphi_U(\tilde{y}_2) = \varphi_U^x(\tilde{y}_2)$, and thus there exists $g_0 \in G_U$ such that $\tilde{y}_1 = g_0(\tilde{y}_2)$. Note that for any $h \in G_U \setminus G_U(\tilde{x})$, $h(\widetilde{O}(\tilde{x})) \cap \widetilde{O}(\tilde{x}) = \emptyset$. We get $g_0 \in G_U(\tilde{x})$ and $[\tilde{y}_1]_{G_U(\tilde{x})} = [\tilde{y}_2]_{G_U(\tilde{x})}$.

Let $A$ be the open subset of $\widetilde{O}(\tilde{x})/G_U(\tilde{x})$ and $\pi_U^x : \widetilde{O}(\tilde{x}) \to \widetilde{O}(\tilde{x})/G_U(\tilde{x})$ be the quotient map. Then $(\pi_U^x)^{-1}(A)$ is open in $\widetilde{O}(\tilde{x})$ (and so in $\widetilde{U}$). Since $\varphi_U : \widetilde{U} \to U$ is an open map, $\varphi_U((\pi_U^x)^{-1}(A))$ is an open subset of $U$. Note that $\varphi_U((\pi_U^x)^{-1}(A)) = \varphi_U^x((\pi_U^x)^{-1}(A)) \subset O(x)$. We get that $\varphi_U^x((\pi_U^x)^{-1}(A))$ is an open subset of $O(x)$. Moreover, $\overline{\varphi_U^x}(A) = \varphi_U^x((\pi_U^x)^{-1}(A))$. It follows that $\overline{\varphi_U^x}$ is open. $\square$

**Remark 1.3.** In Lemma 1.2, if $(\widetilde{U}, G_U, \varphi_U)$ is only a Banach orbifold chart, we cannot derive that $(\widetilde{O}(\tilde{x}), G_U(\tilde{x}), \varphi_U^x = \varphi_U|_{\widetilde{O}(\tilde{x})})$ is also a Banach orbifold chart because $G_U$ effectively acting on $\widetilde{U}$ does not imply $G_U(\tilde{x})$ effectively acting on $\widetilde{O}(\tilde{x})$.

**Lemma 1.4.** *Let $(\widetilde{U}, G_U, \varphi_U)$ be a k-regular Banach orbifold chart. Then for any path connected subset $V \subset U$ and any connected component $\widetilde{V}_c$ of $\varphi_U^{-1}(V)$, the restriction $\varphi_U|_{\widetilde{V}_c} : \widetilde{V}_c \to V$ is surjective. Furthermore, other connected component of $\varphi_U^{-1}(V)$ must be the form of $g(\widetilde{V}_c)$ for $g \in G_U$.*

**Proof.** Choose a regular point $\tilde{x} \in \widetilde{V}_c^{\circ}$. For any $y \in V$, we can choose a continuous path $\ell : [0, 1] \to V$ from $x$ to $y$ such that $\ell([0, 1))$ has only finite intersection points with $V^{\mathrm{sing}}$, i.e., there exist $0 < t_1 < \cdots < t_k < t_{k+1} = 1$ such that

$$\ell(t_i) \in V^{\mathrm{sing}}, \quad \ell\big((t_i, t_{i+1})\big) \subset V^{\circ}, \quad i = 1, \dots, k$$

since $\ell([0, 1])$ is compact and locally $U^{\mathrm{sing}}$ is the union of finite submanifolds of codimension not less than 1. Note that $\varphi_U : \widetilde{U}^{\circ} \to U^{\circ}$ is a $|G_U|$-fold regular covering map. By the properness of $\varphi_U$ and the proof of the uniqueness of the lifting of a path in the covering space theory,[2] it is easy to see that there exists a unique lifting $\tilde{\gamma}_1$ passing through $\tilde{x}$ for $\ell|_{[0,t_1)}$. Obviously, $\tilde{\gamma}_1$ is contained in $\widetilde{V}_c^{\circ}$. Let $\tilde{\gamma}_2$ be a lifting in $\widetilde{U}^{\circ}$ for $\ell|_{(t_1,t_2)}$. From the properness of $\varphi_U$ it follows that the limits $\lim_{t\to t_1-} \tilde{\gamma}_1(t)$ and $\lim_{t\to t_1+} \tilde{\gamma}_2(t)$ exist and

$$\varphi_U\Big(\lim_{t\to t_1-} \tilde{\gamma}_1(t)\Big) = \varphi_U\Big(\lim_{t\to t_1+} \tilde{\gamma}_2(t)\Big).$$

The last equality implies that there exists $g_2 \in G_U$ such that $\lim_{t\to t_1-} \tilde{\gamma}_1(t) = g_2(\lim_{t\to t_1+} \tilde{\gamma}_2(t))$. Define $\tilde{\gamma}_1(t_1) = \lim_{t\to t_1-} \tilde{\gamma}_1(t)$ and $\tilde{\gamma}_2(t_2) = \lim_{t\to t_2-} \tilde{\gamma}_2(t)$. Then by gluing $\tilde{\gamma}_1$ and $g_2 \circ \tilde{\gamma}_2$, we can obtain a lifting for $\ell|_{[0,t_2]}$, denoted by $\tilde{\gamma}_1 \sharp g_2 \circ \tilde{\gamma}_2$. Similarly, we have liftings $\tilde{\gamma}_3, \dots, \tilde{\gamma}_{k+1}$ of $\ell|_{[t_2,t_3]}, \dots, \ell|_{[t_k,1]}$ respectively, and $g_3, \dots, g_{k+1} \in G_U$ such that $\tilde{\gamma}_1 \sharp g_2 \circ \tilde{\gamma}_2$ and $g_3 \circ \tilde{\gamma}_3, \dots, g_{k+1} \circ \tilde{\gamma}_{k+1}$ can be glued into a lifting $\tilde{\gamma}_1 \sharp g_2 \circ \tilde{\gamma}_2 \sharp g_3 \circ \tilde{\gamma}_3 \sharp \cdots \sharp g_{k+1}\tilde{\gamma}_{k+1}$ for $\ell$. Clearly, the lifting must be in $\widetilde{V}_c$ and $\varphi_U(\tilde{\gamma}_1 \sharp g_2 \circ \tilde{\gamma}_2 \sharp g_3 \circ \tilde{\gamma}_3 \sharp \cdots \sharp g_{k+1}\tilde{\gamma}_{k+1}(1)) = y$. The first assertion is proved.

Now we prove the second assertion. Let $\widetilde{V}_1$ be another connected component of $\varphi_U^{-1}(V)$. By the first assertion, we choose $x \in V$, $\tilde{x} \in \widetilde{V}_c$ and $\tilde{x}_1 \in \widetilde{V}_1$ such that $\varphi_U(\tilde{x}) = \varphi_U(\tilde{x}_1)$. Let $g \in G_U$ such that $g(\tilde{x}) = \tilde{x}_1$. Since both $g(\widetilde{V}_c)$ and $g^{-1}(\widetilde{V}_1)$ are connected, we have $g^{-1}(\widetilde{V}_1) \subset \widetilde{V}_c$ and $g(\widetilde{V}_c) \subset \widetilde{V}_1$. Therefore $g(\widetilde{V}_c) = \widetilde{V}_1$. $\square$

Let $(\widetilde{U}, G_U, \varphi_U)$ and $(\widetilde{V}, G_V, \varphi_V)$ be two $k$-regular Banach orbifold charts whose supports $U \subset V \subset X$. We say $\theta_{UV} = (\tilde{\theta}_{UV}, \gamma_{UV})$ to be an *injection* from $(\widetilde{U}, G_U, \varphi_U)$ to $(\widetilde{V}, G_V, \varphi_V)$ if $\gamma_{UV} : G_U \to G_V$ is an injective group homomorphism and $\tilde{\theta}_{UV}$ is a $\gamma_{UV}$-equivariant diffeomorphism from $\widetilde{U}$ onto an open subset of $\widetilde{V}$ such that $\varphi_U = \varphi_V \circ \tilde{\theta}_{UV}$.

**Lemma 1.5.** *Let $\theta_{UV} = (\tilde{\theta}_{UV}, \gamma_{UV})$ be an injection from a k-regular Banach orbifold chart $(\widetilde{U}, G_U, \varphi_U)$ to another $(\widetilde{V}, G_V, \varphi_V)$ as above. If $h \in G_V$ such that $h(\tilde{\theta}_{UV}(\widetilde{U})) \cap \tilde{\theta}_{UV}(\widetilde{U}) \neq \emptyset$, then $h(\tilde{\theta}_{UV}(\widetilde{U})) = \tilde{\theta}_{UV}(\widetilde{U})$ and h sits in the image of homomorphism $\gamma_{UV} : G_U \to G_V$.*

[2] If $(\widetilde{U}, G_U, \varphi_U)$ is 2-regular this directly follows because $\widetilde{U}^{\circ}$ is connected.

**Proof.** Let $C = h(\tilde{\theta}_{UV}(\widetilde{U})) \cap \tilde{\theta}_{UV}(\widetilde{U})$. Since $h : \widetilde{V} \to \widetilde{V}$ is a homeomorphism, it follows that $C$ and $h^{-1}(C)$ are open subsets of $\widetilde{V}$. Note that $C \subset h(\tilde{\theta}_{UV}(\widetilde{U}))$ or $h^{-1}(C) \subset \tilde{\theta}_{UV}(\widetilde{U})$, and that the set of regular points is dense open subset. We can choose a regular point $\tilde{\theta}_{UV}(\tilde{p}) \in h^{-1}(C)$. So the isotropy group $G_V(\tilde{\theta}_{UV}(\tilde{p})) = \{I_{G_V}\}$. Since $h\tilde{\theta}_{UV}(\tilde{p}) \in C$ and $C \subset \tilde{\theta}_{UV}(\widetilde{U})$, we have $\tilde{q} \in \widetilde{U}$ such that $h\tilde{\theta}_{UV}(\tilde{p}) = \tilde{\theta}_{UV}(\tilde{q})$. Hence $\varphi_V \tilde{\theta}_{UV}(\tilde{p}) = \varphi_V h\tilde{\theta}_{UV}(\tilde{p}) = \varphi_V \tilde{\theta}_{UV}(\tilde{q})$, $\varphi_U(\tilde{p}) = \varphi_U(\tilde{q})$. It follows that there exists $g_0 \in G_U$ such that $g_0(\tilde{p}) = \tilde{q}$. Let $\gamma_{UV}(g_0) = h_0$, we get $h\tilde{\theta}_{UV}(\tilde{p}) = \tilde{\theta}_{UV}(\tilde{q}) = \tilde{\theta}_{UV} g_0(\tilde{p}) = h_0\tilde{\theta}_{UV}(\tilde{p})$. Therefore $G_V(\tilde{\theta}_{UV}(\tilde{p})) = \{I_{G_V}\}$ yields $h = h_0$. Finally we have $h(\tilde{\theta}_{UV}(\widetilde{U})) = h_0(\tilde{\theta}_{UV}(\widetilde{U})) = \gamma_{UV}(g_0)(\tilde{\theta}_{UV}(\widetilde{U})) = \tilde{\theta}_{UV}(g_0(\widetilde{U})) = \tilde{\theta}_{UV}(\widetilde{U})$. □

**Remark 1.6.** If $U = V$, then $\tilde{\theta}_{UV}$ is a diffeomorphism from $\widetilde{U}$ to $\widetilde{V}$ and $\gamma_{UV} : G_U \to G_V$ is a group isomorphism. (In this case, we say the two $k$-regular Banach orbifold charts to be *equivalent* and call $\theta_{UV}$ an *equivalence* between them.) In fact, $\varphi_V(\widetilde{V}) = \varphi_U(\widetilde{U}) = \varphi_V(\tilde{\theta}_{UV}(\widetilde{U}))$, hence for any $\tilde{y} \in \widetilde{V}$, we have $\tilde{x} \in \widetilde{U}$ such that $\varphi_V(\tilde{y}) = \varphi_V(\tilde{\theta}_{UV}(\tilde{x}))$. This implies there exists $h \in G_V$ such that $h(\tilde{\theta}_{UV}(\tilde{x})) = \tilde{y}$. So $\widetilde{V} \subset \bigcup_{h \in G_V} h(\tilde{\theta}_{UV}(\widetilde{U}))$. On the other hand, it is easy to see that $\bigcup_{h \in G_V} h(\tilde{\theta}_{UV}(\widetilde{U})) \subset \widetilde{V}$. We thus get $\widetilde{V} = \bigcup_{h \in G_V} h(\tilde{\theta}_{UV}(\widetilde{U}))$. Next we claim that $h(\tilde{\theta}_{UV}(\widetilde{U})) = \tilde{\theta}_{UV}(\widetilde{U})$ for any $h \in G_V$. Otherwise, let us denote

$$\widetilde{V}_1 = \bigcup_{\substack{h \in G_V \\ h(\tilde{\theta}_{UV}(\widetilde{U})) \neq \tilde{\theta}_{UV}(\widetilde{U})}} h\big(\tilde{\theta}_{UV}(\widetilde{U})\big), \qquad \widetilde{V}_2 = \bigcup_{\substack{h \in G_V \\ h(\tilde{\theta}_{UV}(\widetilde{U})) = \tilde{\theta}_{UV}(\widetilde{U})}} h\big(\tilde{\theta}_{UV}(\widetilde{U})\big) = \tilde{\theta}_{UV}(\widetilde{U}),$$

then $\widetilde{V} = \widetilde{V}_1 \cup \widetilde{V}_2$. Since $I_{G_V}(\tilde{\theta}_{UV}(\widetilde{U})) = \tilde{\theta}_{UV}(\widetilde{U})$ for the unit element $I_{G_V}$ of $G_V$, $\widetilde{V}_2 \neq \emptyset$. Note that we also have $\widetilde{V}_1 \cap \widetilde{V}_2 = \emptyset$. Indeed, suppose that there exists $h \in G_V$, $h(\tilde{\theta}_{UV}(\widetilde{U})) \neq \tilde{\theta}_{UV}(\widetilde{U})$ such that $h(\tilde{\theta}_{UV}(\widetilde{U})) \cap \tilde{\theta}_{UV}(\widetilde{U}) \neq \emptyset$. By Lemma 1.5, $h(\tilde{\theta}_{UV}(\widetilde{U})) = \tilde{\theta}_{UV}(\widetilde{U})$, which contradicts $h(\tilde{\theta}_{UV}(\widetilde{U})) \neq \tilde{\theta}_{UV}(\widetilde{U})$. Hence we have proved that $\widetilde{V} = \widetilde{V}_1 \cup \widetilde{V}_2$ and $\widetilde{V}_1 \cap \widetilde{V}_2 = \emptyset$. It contradicts the connectedness of $\widetilde{V}$. So $\widetilde{V} = \tilde{\theta}_{UV}(\widetilde{U})$. By Lemma 1.5 again, $h$ is in the image of $\gamma_{UV}$ and $\gamma_{UV}$ is an isomorphism.

**Definition 1.7.** Two $k$-regular Banach orbifold charts $(\widetilde{U}_\alpha, G_\alpha, \varphi_\alpha)$ and $(\widetilde{U}_\beta, G_\beta, \varphi_\beta)$ with supports in Hausdorff topological space $X$ are called *compatible*, if either $U_\alpha \cap U_\beta = \emptyset$, or $U_\alpha \cap U_\beta \neq \emptyset$, but for any $x \in U_\alpha \cap U_\beta$, $\tilde{x}_\alpha \in (\varphi_\alpha)^{-1}(x)$ and $\tilde{x}_\beta \in (\varphi_\beta)^{-1}(x)$ there exist equivalent induced charts $(\widetilde{O}(\tilde{x}_\alpha), G_\alpha(\tilde{x}_\alpha), \varphi_\alpha^x)$ at $\tilde{x}_\alpha$ and $(\widetilde{O}(\tilde{x}_\beta), G_\beta(\tilde{x}_\beta), \varphi_\beta^x)$ at $\tilde{x}_\beta$. A family of $k$-regular Banach orbifold charts $\mathcal{A} = \{(\widetilde{U}_\alpha, G_\alpha, \varphi_\alpha)\}_{\alpha \in \Lambda}$ is called a *$k$-regular Banach orbifold atlas* if any two charts in it are compatible and $\bigcup_{\alpha \in \Lambda} U_\alpha = X$.

Let $\overline{\mathcal{A}}$ be the set of all $k$-regular Banach orbifold charts with supports in $X$ which are compatible with any chart in $\mathcal{A}$. Obviously, it is uniquely determined by $\mathcal{A}$. We say $\overline{\mathcal{A}}$ to be a *$k$-regular Banach orbifold structure* on $X$. Two $k$-regular Banach orbifold atlases $\mathcal{A}$ and $\mathcal{A}'$ on $X$ are called *equivalent* if $\mathcal{A} \cup \mathcal{A}'$ is a $k$-regular Banach orbifold atlas. It is easily checked that two equivalent $k$-regular Banach orbifold atlases determine the same $k$-regular Banach orbifold structure. A Hausdorff topological space $X$ together with a $k$-regular Banach orbifold structure $\overline{\mathcal{A}}$ on it is called a *$k$-regular Banach orbifold*, denoted by $(X, \overline{\mathcal{A}})$. Later when we refer to a $k$-regular Banach orbifold, we assume it has been given a $k$-regular Banach orbifold atlas $\mathcal{A}$, denoted by $(X, \mathcal{A})$ and simplified by $X$ without occurring of confusions.

**Remark 1.8.** According to Satake's definition in [2], one should define a $k$-regular Banach orbifold atlas to be a family of $k$-regular Banach orbifold charts $\mathcal{A} = \{(\widetilde{U}_\alpha, G_\alpha, \varphi_\alpha)\}_{\alpha \in \Lambda}$ satisfying:

(i) $X = \bigcup_{\alpha\in\Lambda} U_\alpha$,
(ii) for any charts $(\widetilde{U}_\alpha, G_\alpha, \varphi_\alpha)$ and $(\widetilde{U}_\beta, G_\beta, \varphi_\beta)$ in $\mathcal{A}$, and $x \in U_\alpha \cap U_\beta$, there exists a chart $(\widetilde{U}_\gamma, G_\gamma, \varphi_\gamma)$ in $\mathcal{A}$ such that $x \in U_\gamma \subset U_\alpha \cap U_\beta$,
(iii) if the support $U_\alpha$ of $(\widetilde{U}_\alpha, G_\alpha, \varphi_\alpha)$ is contained in the support $U_\beta$ of $(\widetilde{U}_\beta, G_\beta, \varphi_\beta)$, there exists an injection from $(\widetilde{U}_\alpha, G_\alpha, \varphi_\alpha)$ to $(\widetilde{U}_\beta, G_\beta, \varphi_\beta)$.

Our definition above is an infinitely dimensional version of the definition of finite-dimensional orbifolds in [6, Chapter 2], which is convenient for our arguments in this paper. If $X$ is paracompact, we can prove that the two definitions are equivalent.

For a nonempty open subset $W$ in a $k$-regular Banach orbifold $(X, \mathcal{A})$, it follows from Lemma 1.4 that the atlas $\mathcal{A} = \{(\widetilde{U}_\alpha, G_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ naturally induces a $k$-regular Banach orbifold atlas $\mathcal{A}_W$ on $W$, which consists of the following charts

$$\bigl(\widetilde{V}_\alpha^{c0}, G_\alpha^{c0}, \varphi_\alpha^{c0}\bigr),$$

where $(\widetilde{U}_\alpha, G_\alpha, \varphi_\alpha)$ is any chart in $\mathcal{A}$, $(W \cap U_\alpha)_c$ is a path connected component of $W \cap U_\alpha$ and $\widetilde{V}_\alpha^{c0}$ is a connected component of $\varphi_\alpha^{-1}((W \cap U_\alpha)_c)$, $G_\alpha^{c0}$ is the subgroup of $G_\alpha$ fixing $\widetilde{V}_\alpha^{c0}$, $\varphi_\alpha^{c0}$ is the restriction of $\varphi_\alpha$ to $\widetilde{V}_\alpha^{c0}$.

**Definition 1.9.** Let $(X, \mathcal{A}_X)$ and $(Y, \mathcal{A}_Y)$ be 1-regular Banach orbifolds, $\mathcal{A}_X = \{(\widetilde{U}_\alpha, G_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$, $\mathcal{A}_Y = \{(\widetilde{W}_k, H_k, \psi_k)\}_{k\in\Gamma}$. A continuous map $f : X \to Y$ is called a $C^\infty$*-orbifold map* if for each chart $(\widetilde{U}_\alpha, G_\alpha, \varphi_\alpha)$ in $\mathcal{A}_X$ there exists a chart $(\widetilde{W}_{k(\alpha)}, H_{k(\alpha)}, \psi_{k(\alpha)})$ in $\mathcal{A}_Y$ such that $f(U_\alpha) \subset W_{k(\alpha)}$ and there exist a $C^\infty$ map $\tilde{f}_\alpha : \widetilde{U}_\alpha \to \widetilde{W}_{k(\alpha)}$ and a group homomorphism $\tau_\alpha : G_\alpha \to H_{k(\alpha)}$ satisfying $\tilde{f}_\alpha \circ g = \tau_\alpha(g) \circ \tilde{f}_\alpha$ and $\psi_{k(\alpha)} \circ \tilde{f}_\alpha = f \circ \varphi_\alpha$. In this case, $\tilde{f}_\alpha$ is called a lifting of $f : U_\alpha \to W_\alpha$, and $(\tilde{f}_\alpha, \tau_\alpha) : (\widetilde{U}_\alpha, G_\alpha, \varphi_\alpha) \to (\widetilde{W}_{k(\alpha)}, H_{k(\alpha)}, \psi_{k(\alpha)})$ is called a *local representation* of $f$ near any $x \in U_\alpha$. Later, $k(\alpha)$ is denoted by $\alpha$ for the sake of simplicity.

Later, when we refer to a orbifold map $f : X \to Y$, we always assume that it is respect to the given atlases $\mathcal{A}_X$ and $\mathcal{A}_Y$. Now let $f : X \to Y$ be a $C^\infty$-orbifold map from a 2-regular Banach orbifold $(X, \mathcal{A}_X)$ to a 1-regular Banach orbifold $(Y, \mathcal{A}_Y)$. Suppose that $(\tilde{f}_\alpha, \tau_\alpha) : (\widetilde{U}_\alpha, G_\alpha, \varphi_\alpha) \to (\widetilde{W}_\alpha, H_\alpha, \psi_\alpha)$ is a local representation of $f$ near $x \in U_\alpha$. If for any $\tilde{x}_\alpha \in (\varphi_\alpha)^{-1}(x)$, the tangent mapping $d\tilde{f}_\alpha(\tilde{x}_\alpha) : T_{\tilde{x}_\alpha}\widetilde{U}_\alpha \to T_{\tilde{f}_\alpha(\tilde{x}_\alpha)}\widetilde{W}_\alpha$ is a linear topological isomorphism and $\tau_\alpha|_{G_\alpha(\tilde{x}_\alpha)} : G_\alpha(\tilde{x}_\alpha) \to H_\alpha(\tilde{f}_\alpha(\tilde{x}_\alpha))$ is surjective, we say $f$ to be *normal* at $x$. The following lemma shows that this definition is well defined, i.e., does not depend on the choice of the point $\tilde{x}_\alpha \in (\varphi_\alpha)^{-1}(x)$, nor on the chart $(\widetilde{U}_\alpha, G_\alpha, \varphi_\alpha)$ containing $x$.

**Lemma 1.10.** *Let $(X, \mathcal{A}_X)$ be a 2-regular Banach orbifold and $(Y, \mathcal{A}_Y)$ be a 1-regular Banach orbifold, $\mathcal{A}_X = \{(\widetilde{U}_\alpha, G_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$, $f : X \to Y$ be a $C^\infty$-orbifold map. Suppose that $(\tilde{f}_\alpha, \tau_\alpha) : (\widetilde{U}_\alpha, G_\alpha, \varphi_\alpha) \to (\widetilde{W}_\alpha, H_\alpha, \psi_\alpha)$ and $(\tilde{f}_\beta, \tau_\beta) : (\widetilde{U}_\beta, G_\beta, \varphi_\beta) \to (\widetilde{W}_\beta, H_\beta, \psi_\beta)$ are two local representation of $f$. For any $x \in U_\alpha \cap U_\beta$, $\tilde{x}_\alpha \in (\varphi_\alpha)^{-1}(x)$ and $\tilde{x}_\beta \in (\varphi_\beta)^{-1}(x)$, if the tangent map $d\tilde{f}_\alpha(\tilde{x}_\alpha)$ is a linear topological isomorphism and $\tau_\alpha|_{G_\alpha(\tilde{x}_\alpha)} : G_\alpha(\tilde{x}_\alpha) \to H_\alpha(\tilde{f}_\alpha(\tilde{x}_\alpha))$ is surjective, then the tangent map $d\tilde{f}_\beta(\tilde{x}_\beta)$ is a linear topological isomorphism and $\tau_\beta|_{G_\beta(\tilde{x}_\beta)} : G_\beta(\tilde{x}_\beta) \to H_\beta(\tilde{f}_\beta(\tilde{x}_\beta))$ is surjective too.*

**Proof.** Since $x \in U_\alpha \cap U_\beta$, we have $f(x) \in W_\alpha \cap W_\beta$, $\tilde{f}_\alpha(\tilde{x}_\alpha) \in (\psi_\alpha)^{-1}(f(x))$ and $\tilde{f}_\beta(\tilde{x}_\beta) \in (\psi_\beta)^{-1}(f(x))$. By the definition of orbifolds, there are two equivalent induced charts

$$\big(\widetilde{O}(\tilde{x}_\alpha), G_\alpha(\tilde{x}_\alpha), \varphi_\alpha^x\big) \quad \text{and} \quad \big(\widetilde{O}(\tilde{x}_\beta), G_\beta(\tilde{x}_\beta), \varphi_\beta^x\big)$$

at $x$ and an equivalence between them

$$\big(\lambda_{\alpha\beta}^U, \mathcal{A}_{\alpha\beta}^U\big): \big(\widetilde{O}(\tilde{x}_\alpha), G_\alpha(\tilde{x}_\alpha), \varphi_\alpha^x\big) \to \big(\widetilde{O}(\tilde{x}_\beta), G_\beta(\tilde{x}_\beta), \varphi_\beta^x\big)$$

such that $\lambda_{\alpha\beta}^U(\tilde{x}_\alpha) = \tilde{x}_\beta$. Similarly, there are two equivalent induced charts

$$\big(\widetilde{O}\big(\tilde{f}_\alpha(\tilde{x}_\alpha)\big), H_\alpha\big(\tilde{f}_\alpha(\tilde{x}_\alpha)\big), \psi_\alpha^{f(x)}\big) \quad \text{and} \quad \big(\widetilde{O}\big(\tilde{f}_\beta(\tilde{x}_\beta)\big), H_\beta\big(\tilde{f}_\beta(\tilde{x}_\beta)\big), \psi_\beta^{f(x)}\big)$$

at $f(x)$ and an equivalence between them

$$\big(\lambda_{\alpha\beta}^W, \mathcal{A}_{\alpha\beta}^W\big): \big(\widetilde{O}\big(\tilde{f}_\alpha(\tilde{x}_\alpha)\big), H_\alpha\big(\tilde{f}_\alpha(\tilde{x}_\alpha)\big), \psi_\alpha^{f(x)}\big) \to \big(\widetilde{O}\big(\tilde{f}_\beta(\tilde{x}_\beta)\big), H_\beta\big(\tilde{f}_\beta(\tilde{x}_\beta)\big), \psi_\beta^{f(x)}\big)$$

such that $\lambda_{\alpha\beta}^W(\tilde{f}_\alpha(\tilde{x}_\alpha)) = \tilde{f}_\beta(\tilde{x}_\beta)$. We can choose $\widetilde{O}(\tilde{x}_\alpha)$ and $\widetilde{O}(\tilde{x}_\beta)$ so small that $\tilde{f}_\alpha(\widetilde{O}(\tilde{x}_\alpha)) \subset \widetilde{O}(\tilde{f}_\alpha(\tilde{x}_\alpha))$ and $\tilde{f}_\beta(\widetilde{O}(\tilde{x}_\beta)) \subset \widetilde{O}(\tilde{f}_\beta(\tilde{x}_\beta))$. Using the facts that $\varphi_\alpha^x = \varphi_\beta^x \circ \lambda_{\alpha\beta}^U$, $\psi_\alpha^{f(x)} = \psi_\beta^{f(x)} \circ \lambda_{\alpha\beta}^W$ and $\psi_\alpha^{f(x)} \circ \tilde{f}_\alpha = f \circ \varphi_\alpha^x$, $\psi_\beta^{f(x)} \circ \tilde{f}_\beta = f \circ \varphi_\beta^x$, we have

$$\psi_\beta^{f(x)} \circ \lambda_{\alpha\beta}^W \circ \tilde{f}_\alpha = \psi_\alpha^{f(x)} \circ \tilde{f}_\alpha = f \circ \varphi_\alpha^x = f \circ \varphi_\beta^x \circ \lambda_{\alpha\beta}^U = \psi_\beta^{f(x)} \circ \tilde{f}_\beta \circ \lambda_{\alpha\beta}^U. \tag{2}$$

Since $d\tilde{f}_\alpha(\tilde{x}_\alpha): T_{\tilde{x}_\alpha}\widetilde{U}_\alpha \to T_{\tilde{f}_\alpha(\tilde{x}_\alpha)}\widetilde{W}_\alpha$ is a linear topological isomorphism, by the usual inverse function theorem we can assume $\tilde{f}_\alpha|_{\widetilde{O}(\tilde{x}_\alpha)}$ is a homeomorphism (shrinking $\widetilde{O}(\tilde{x}_\alpha)$ if necessary). For any connected open subset $\widetilde{O}(\tilde{f}_\alpha(\tilde{x}_\alpha))$ of $\tilde{f}_\alpha(\widetilde{O}(\tilde{x}_\alpha))$, we claim: $\tilde{f}_\alpha(\widetilde{O}(\tilde{x}_\alpha))$ *is* $H_\alpha(\tilde{f}_\alpha(\tilde{x}_\alpha))$*-invariant*. Indeed, since $\tau_\alpha|_{G_\alpha(\tilde{x}_\alpha)}: G_\alpha(\tilde{x}_\alpha) \to H_\alpha(\tilde{f}_\alpha(\tilde{x}_\alpha))$ is surjective, for any $h_\alpha \in H_\alpha(\tilde{f}_\alpha(\tilde{x}_\alpha))$, there exists a $g_\alpha \in G_\alpha(\tilde{x}_\alpha)$ such that $\tau_\alpha|_{G_\alpha(\tilde{x}_\alpha)}(g_\alpha) = h_\alpha$. For any $\tilde{x} \in \widetilde{O}(\tilde{x}_\alpha)$, $\tilde{y} = \tilde{f}_\alpha(\tilde{x}) \in \tilde{f}_\alpha(\widetilde{O}(\tilde{x}_\alpha))$, we arrive at

$$h_\alpha(\tilde{y}) = \tau_\alpha|_{G_\alpha(\tilde{x}_\alpha)}(g_\alpha) \circ \tilde{f}_\alpha(\tilde{x}) = \tilde{f}_\alpha \circ g_\alpha(\tilde{x}) \in \tilde{f}_\alpha\big(\widetilde{O}(\tilde{x}_\alpha)\big).$$

On the other hand, for any $h \in H_\alpha \setminus H_\alpha(\tilde{f}_\alpha(\tilde{x}_\alpha))$, it follows from $h(\widetilde{O}((\tilde{f}_\alpha(\tilde{x}_\alpha))) \cap \widetilde{O}((\tilde{f}_\alpha(\tilde{x}_\alpha))) = \emptyset$ and $\tilde{f}_\alpha(\widetilde{O}(\tilde{x}_\alpha)) \subset \widetilde{O}(\tilde{f}_\alpha(\tilde{x}_\alpha))$ that $h(\tilde{f}_\alpha(\widetilde{O}(\tilde{x}_\alpha))) \cap \tilde{f}_\alpha(\widetilde{O}(\tilde{x}_\alpha)) = \emptyset$. Denote by

$$H_\alpha^0 = \big\{h_\alpha|_{\tilde{f}_\alpha(\widetilde{O}(\tilde{x}_\alpha))} \bigm| h_\alpha \in H_\alpha\big(\tilde{f}_\alpha(\tilde{x}_\alpha)\big)\big\}.$$

We get a 1-regular Banach orbifold chart $(\tilde{f}_\alpha(\widetilde{O}(\tilde{x}_\alpha)), H_\alpha^0, \psi_\alpha^0 = \psi_\alpha|_{\tilde{f}_\alpha(\widetilde{O}(\tilde{x}_\alpha))})$.

We claim that the map $\tilde{f}_\alpha: \widetilde{O}(\tilde{x}_\alpha) \to \tilde{f}_\alpha(\widetilde{O}(\tilde{x}_\alpha))$ sends the regular points in $(\widetilde{O}(\tilde{x}_\alpha), G_\alpha(x), \varphi_\alpha^x)$ to the regular points in $(\tilde{f}_\alpha(\widetilde{O}(\tilde{x}_\alpha)), H_\alpha^0, \psi_\alpha^0)$. In fact, let $\tilde{x} \in \widetilde{O}(\tilde{x}_\alpha)$ be a regular point. For any $h_\alpha \in H_\alpha^0 \setminus \{I\}$, since the map $\tau_\alpha|_{G_\alpha(\tilde{x}_\alpha)}: G_\alpha(\tilde{x}_\alpha) \to H_\alpha(\tilde{f}_\alpha(\tilde{x}_\alpha))$ is surjective there exists $g_\alpha \in G_\alpha(\tilde{x}_\alpha) \setminus \{I\}$ such that $\tau_\alpha|_{G_\alpha(\tilde{x}_\alpha)}(g_\alpha) = h_\alpha$. It follows that

$$h_\alpha \circ \tilde{f}_\alpha(\tilde{x}) = \tau_\alpha|_{G_\alpha(\tilde{x}_\alpha)}(g_\alpha) \circ \tilde{f}_\alpha(\tilde{x}) = \tilde{f}_\alpha \circ g_\alpha(\tilde{x}). \tag{3}$$

Note that $\tilde{x}$ is a regular point. We have $g_\alpha(\tilde{x}) \neq \tilde{x}$. Moreover, since $\tilde{f}_\alpha|_{\widetilde{O}(\tilde{x}_\alpha)}$ is a homeomorphism, we get $\tilde{f}_\alpha \circ g_\alpha(\tilde{x}) \neq \tilde{f}_\alpha(\tilde{x})$. Combining this with (3) we arrive at $h_\alpha \circ \tilde{f}_\alpha(\tilde{x}) \neq \tilde{f}_\alpha(\tilde{x})$ for any $h_\alpha \in H_\alpha^0 \setminus \{I\}$. It means that $\tilde{f}_\alpha(\tilde{x})$ is a regular point.

Denote by $H_\beta^0 = \{h_\beta|_{\lambda_{\alpha\beta}^W \circ \tilde{f}_\alpha(\widetilde{O}(\tilde{x}_\alpha))} \mid h_\beta \in H_\beta(\tilde{f}_\beta(\tilde{x}_\beta))\}$ and $\psi_\beta^0 = \psi_\beta|_{\lambda_{\alpha\beta}^W \circ \tilde{f}_\alpha(\widetilde{O}(\tilde{x}_\alpha))}$. Then $(\lambda_{\alpha\beta}^W \circ \tilde{f}_\alpha(\widetilde{O}(\tilde{x}_\alpha)), H_\beta^0, \psi_\beta^0)$ is also a 1-regular Banach orbifold chart. Since the equivalence

$(\lambda^W_{\alpha\beta}, \mathcal{A}^W_{\alpha\beta})$ always maps the regular points in $(\widetilde{O}(\tilde{f}_\alpha(\tilde{x}_\alpha)), H_\alpha(\tilde{f}_\alpha(\tilde{x}_\alpha)), \psi_\alpha^{f(x)})$ to the regular points in $(\widetilde{O}(\tilde{f}_\beta(\tilde{x}_\beta)), H_\beta(\tilde{f}_\beta(\tilde{x}_\beta)), \psi_\beta^{f(x)})$, the restriction $\lambda^W_{\alpha\beta}|_{\tilde{f}_\alpha(\widetilde{O}(\tilde{x}_\alpha))} : \tilde{f}_\alpha(\widetilde{O}(\tilde{x}_\alpha)) \to \lambda^W_{\alpha\beta} \circ \tilde{f}_\alpha(\widetilde{O}(\tilde{x}_\alpha))$ also maps the regular points in $(\tilde{f}_\alpha(\widetilde{O}(\tilde{x}_\alpha)), H^0_\alpha, \psi^0_\alpha)$ to the regular points in $(\lambda^W_{\alpha\beta} \circ \tilde{f}_\alpha(\widetilde{O}(\tilde{x}_\alpha)), H^0_\beta, \psi^0_\beta)$.

Choose a regular point $\tilde{x} \in \widetilde{O}(\tilde{x}_\alpha)$. The arguments above shows that $\lambda^W_{\alpha\beta} \circ \tilde{f}_\alpha(\tilde{x}) \in \lambda^W_{\alpha\beta} \circ \tilde{f}_\alpha(\widetilde{O}(\tilde{x}_\alpha))$ is also a regular point. By (2), we get

$$\psi_\beta^{f(x)} \circ \lambda^W_{\alpha\beta} \circ \tilde{f}_\alpha(\tilde{x}) = \psi_\beta^{f(x)} \circ \tilde{f}_\beta \circ \lambda^U_{\alpha\beta}(\tilde{x}).$$

Hence there exists a uniquely determined $h_\beta(\tilde{x}) \in H^0_\beta$ such that $h_\beta(\tilde{x}) \circ \lambda^W_{\alpha\beta} \circ \tilde{f}_\alpha(\tilde{x}) = \tilde{f}_\beta \circ \lambda^U_{\alpha\beta}(\tilde{x})$. We claim that there exists $h_0 \in H^0_\beta$ such that

$$h_0 \circ \lambda^W_{\alpha\beta} \circ \tilde{f}_\alpha(\tilde{y}) = \tilde{f}_\beta \circ \lambda^U_{\alpha\beta}(\tilde{y}) \quad \forall \tilde{y} \in \widetilde{O}(\tilde{x}_\alpha)^\circ. \tag{4}$$

Let $h^i_\beta$, $i = 1, 2, \dots, n$, be all elements of $H^0_\beta$. Denote by $\widetilde{O}_i = \{\tilde{x} \in \widetilde{O}(\tilde{x}_\alpha)^\circ \mid h^i_\beta \circ \lambda^W_{\alpha\beta} \circ \tilde{f}_\alpha(\tilde{x}) = \tilde{f}_\beta \circ \lambda^U_{\alpha\beta}(\tilde{x})\}$, $i = 1, \dots, n$. If the claim is not true, then any $\widetilde{O}_i \neq \widetilde{O}(\tilde{x}_\alpha)^\circ$, $i = 1, \dots, n$, which imply there are, at least, two $\widetilde{O}_i \neq \emptyset$. Note that as the relative closed set of $\widetilde{O}(\tilde{x}_\alpha)^\circ$, any two of $\widetilde{O}_1, \dots, \widetilde{O}_n$ are not intersecting. (Indeed, if $\tilde{x} \in \widetilde{O}_i \cap \widetilde{O}_j$ for some $i \neq j$, then $h^i_\beta \circ \lambda^W_{\alpha\beta} \circ \tilde{f}_\alpha(\tilde{x}) = h^j_\beta \circ \lambda^W_{\alpha\beta} \circ \tilde{f}_\alpha(\tilde{x})$, which is a contradiction with the uniqueness of $h_\beta(\tilde{x})$.) Hence $\widetilde{O}(\tilde{x}_\alpha)^\circ$ is a union of at least two nonempty disjoint closed subsets. Therefore it is not connected. On the other hand, the chart $(\widetilde{O}(\tilde{x}_\alpha), G_\alpha(x), \varphi^x_\alpha)$ is 2-regular, one easily uses (1) to prove $\widetilde{O}(\tilde{x}_\alpha)^\circ = \widetilde{O}(\tilde{x}_\alpha) \setminus \widetilde{O}(\tilde{x}_\alpha)^{\mathrm{sing}}$ is connected. This contradiction shows that (4) holds.

Finally, since $\widetilde{O}(\tilde{x}_\alpha)^\circ$ is a dense open subset of $\widetilde{O}(\tilde{x}_\alpha)$, we arrive at $h_0 \circ \lambda^W_{\alpha\beta} \circ \tilde{f}_\alpha = \tilde{f}_\beta \circ \lambda^U_{\alpha\beta}$ on $\widetilde{O}(\tilde{x}_\alpha)$. In particular, we have $h_0 \circ \lambda^W_{\alpha\beta} \circ \tilde{f}_\alpha(\tilde{x}_\alpha) = \tilde{f}_\beta \circ \lambda^U_{\alpha\beta}(\tilde{x}_\alpha) = \tilde{f}_\beta(\tilde{x}_\beta)$. Notice that $h_0 \circ \lambda^W_{\alpha\beta}$ is a diffeomorphism and that $d(h_0 \circ \lambda^W_{\alpha\beta})(\tilde{f}_\alpha(\tilde{x}_\alpha)) \circ d\tilde{f}_\alpha(\tilde{x}_\alpha) = d\tilde{f}_\beta(\tilde{x}_\beta)$. We obtain $d\tilde{f}_\beta(\tilde{x}_\beta) : T_{\tilde{x}_\beta}\widetilde{U}_\beta \to T_{\tilde{f}_\beta(\tilde{x}_\beta)}\widetilde{W}_\beta$ is a linear topological isomorphism.

Moreover, since both $\mathcal{A}^U_{\alpha\beta} : G_\alpha(\tilde{x}_\alpha) \to G_\beta(\tilde{x}_\beta)$ and $\mathcal{A}^W_{\alpha\beta} : H_\alpha(\tilde{f}_\alpha(\tilde{x}_\alpha)) \to H_\beta(\tilde{f}_\beta(\tilde{x}_\beta))$ are isomorphisms, and $\tau_\alpha|_{G_\alpha(\tilde{x}_\alpha)} : G_\alpha(\tilde{x}_\alpha) \to H_\alpha(\tilde{f}_\alpha(\tilde{x}_\alpha))$ is surjective, we get that $\tau_\beta|_{G_\beta(\tilde{x}_\beta)} : G_\beta(\tilde{x}_\beta) \to H_\beta(\tilde{f}_\beta(\tilde{x}_\beta))$ is surjective too. □

## 2. Local inverse theorem

**Theorem 2.1.** *Let $(X, \mathcal{A}_X)$ be a 2-regular Banach orbifold and $(Y, \mathcal{A}_Y)$ be a 1-regular Banach orbifold, $\mathcal{A}_X = \{(\widetilde{U}_\alpha, G_\alpha, \varphi_\alpha)\}_{\alpha \in \Lambda}$. Suppose that a $C^\infty$-orbifold map $f : X \to Y$ is normal at $x \in X$. Then there exists an open subset $O \subset U$ of $x$ such that $f$ is invertible on $O$, and the inverse of $f|_O$ is also a $C^\infty$-orbifold map.*

**Proof.** Choose $(\widetilde{U}, G, \varphi) \in \mathcal{A}_X$ and $(\widetilde{W}, H, \psi) \in \mathcal{A}_Y$ such that $f(U) \subset W$. Thus we have a lifting $\tilde{f} : \widetilde{U} \to \widetilde{W}$ of $f|_U$ and a group homomorphism $\tau : G \to H$ such that $\tilde{f} \circ g = \tau(g) \circ \tilde{f}$ and $\psi \circ \tilde{f} = f \circ \varphi$. Moreover, since $f$ is normal at $x$, for any $\tilde{x} \in \varphi^{-1}(x)$, $\tau_{\tilde{x}} = \tau|_{G(\tilde{x})} : G(\tilde{x}) \to H(f(\tilde{x}))$ is surjective.

By Lemma 1.2, we can choose open neighborhoods $\widetilde{O} \subset \widetilde{U}$ of $\tilde{x}$ and $\widetilde{O}(\tilde{f}(\tilde{x})) \subset \widetilde{W}$ of $\tilde{f}(\tilde{x})$ such that $(\widetilde{O}, G(\tilde{x}), \varphi^x = \varphi|_{\widetilde{O}})$ and $(\widetilde{O}(\tilde{f}(\tilde{x})), H(\tilde{f}(\tilde{x})), \psi^{f(x)} = \psi|_{\widetilde{O}(\tilde{f}(\tilde{x}))})$ are 2-regular and 1-regular Banach orbifold charts respectively.

Now for the lifting $\tilde{f}:\widetilde{U}\to\widetilde{W}$ and $\tilde{x}\in\varphi^{-1}(x)$, the tangent map $d\tilde{f}(\tilde{x}):T_{\tilde{x}}\widetilde{U}\to T_{\tilde{f}(\tilde{x})}\widetilde{W}$ is a linear topological isomorphism. We can also assume $\tilde{f}|_{\widetilde{O}}$ is a homeomorphism (shrinking $\widetilde{O}$ if necessary). For the connected open subset $\tilde{f}(\widetilde{O})$ of $\widetilde{O}(\tilde{f}(\tilde{x}))$, we get that $\tilde{f}(\widetilde{O})$ is $H(\tilde{f}(\tilde{x}))$-invariant and $h(\tilde{f}(\widetilde{O}))\cap\tilde{f}(\widetilde{O})=\emptyset$ for any $h\in H\setminus H(\tilde{f}(\tilde{x}))$. (The proof can be obtained by repeating some relevant arguments in Lemma 1.10.) Thus $(\tilde{f}(\widetilde{O}),H(\tilde{f}(\tilde{x})),\psi^{f(x)}=\psi|_{\tilde{f}(\widetilde{O})})$ is also a 1-regular Banach orbifold chart. Since $\tilde{f}:\widetilde{O}\to\tilde{f}(\widetilde{O})$ and $\tau_{\tilde{x}}:G(\tilde{x})\to H(\tilde{f}(\tilde{x}))$ satisfy $\tilde{f}\circ g=\tau_{\tilde{x}}(g)\circ\tilde{f}$ and $\psi^{f(x)}\circ\tilde{f}=f\circ\varphi^x$, it follows that $f|_O:O\to f(O)$ is a $C^\infty$-orbifold map with respect to the orbifold atlases $\{(\widetilde{O},G(\tilde{x}),\varphi^x)\}$ and $\{\tilde{f}(\widetilde{O}),H(\tilde{f}(\tilde{x})),\psi^{f(x)}\}$.

We claim $f$ is invertible on $O=\varphi(\widetilde{O})$ and the inverse is also a $C^\infty$-orbifold map.

**Step 1.** *$f$ is invertible on $O$.* Let $x,y\in O$, $f(x)=f(y)$. For $\tilde{x}\in(\varphi^x)^{-1}(x)\in\widetilde{O}$ and $\tilde{y}\in(\varphi^x)^{-1}(y)\in\widetilde{O}$, we have $\psi^{f(x)}\circ\tilde{f}(\tilde{x})=f\circ\varphi^x(\tilde{x})=f(x)=f(y)=f\circ\varphi^x(\tilde{y})=\psi^{f(x)}\circ\tilde{f}(\tilde{y})$. It follows that there exists $h\in H(\tilde{f}(\tilde{x}))$ such that $\tilde{f}(\tilde{x})=h\circ\tilde{f}(\tilde{y})$. Since $\tau|_{G(\tilde{x})}:G(\tilde{x})\to H(f(\tilde{x}))$ is surjective, there exists $g\in G(\tilde{x})$ such that $\tau_{\tilde{x}}(g)=h$. Moreover, $g(\tilde{y})\in\widetilde{O}$, $\tilde{f}(\tilde{x})=h\circ\tilde{f}(\tilde{y})=\tilde{f}\circ g(\tilde{y})$, and $\tilde{f}|_{\widetilde{O}}$ is a homeomorphism, we derive $\tilde{x}=g(\tilde{y})$ and thus $x=\varphi^x(\tilde{x})=\varphi^x\circ g(\tilde{y})=\varphi^x(\tilde{y})=y$.

**Step 2.** *$(f|_O)^{-1}:f(O)\to O$ is continuous.* For any open subset $A\subset O$, we shall show that $f(A)$ is also open in $f(O)$. Note that $f(A)=\psi^{f(x)}\circ\tilde{f}\circ(\varphi^x)^{-1}(A)$ and that $\tilde{f}|_{\widetilde{O}}$ is a homeomorphism. We get that $\tilde{f}\circ(\varphi^x)^{-1}(A)$ is an open subset of $\tilde{f}(\widetilde{O})$. Finally, since $\psi^{f(x)}:\tilde{f}(\widetilde{O})\to f(O)$ is an open map, we get $f(A)$ is open in $f(O)$.

**Step 3.** *$(f|_O)^{-1}:f(O)\to O$ is a $C^\infty$-orbifold map.* Obviously, $(f|_O)^{-1}:f(O)\to O$, $\tilde{f}^{-1}:\tilde{f}(\widetilde{O})\to\widetilde{O}$ and $(\tau_{\tilde{x}})^{-1}:H(\tilde{f}(\tilde{x}))\to G(\tilde{x})$ satisfy:

(i) $\tilde{f}^{-1}\circ h(\zeta)=\tilde{f}^{-1}\circ h\circ\tilde{f}\circ\tilde{f}^{-1}(\zeta)=\tilde{f}^{-1}\circ\tilde{f}\circ(\tau_{\tilde{x}})^{-1}(h)\circ\tilde{f}^{-1}(\zeta)=(\tau_{\tilde{x}})^{-1}(h)\circ\tilde{f}^{-1}(\zeta)$ for any $h\in H(\tilde{f}(\tilde{x}))$ and $\zeta\in\tilde{f}(\widetilde{O})$.

(ii) $\varphi^x\circ\tilde{f}^{-1}(\zeta)=f^{-1}\circ\psi^{f(x)}(\zeta)$ for any $\zeta\in\tilde{f}(\widetilde{O})$. Indeed, let $\tilde{x}=\tilde{f}^{-1}(\zeta)$. Since $f|_O$ is an orbifold map, we have $f\circ\varphi^x\circ\tilde{f}^{-1}\circ\tilde{f}(\tilde{x})=f\circ\varphi^x(\tilde{x})=\psi^{f(x)}\circ\tilde{f}(\tilde{x})$ and $f\circ\varphi^x\circ\tilde{f}^{-1}(\zeta)=\psi^{f(x)}(\zeta)$. Note that $f|_O$ is invertible. We get $\varphi^x\circ\tilde{f}^{-1}(\zeta)=f^{-1}\circ\psi^{f(x)}(\zeta)$. □

**Remark 2.2.** It is easy to see that the orbifolds in the sense of [1, Definition 2.4] are 2-regular Banach orbifolds. Moreover, [1, Lemma 4.4] shows that the second hypothesis of [1, Theorem 4.3] implies $\tau_{\tilde{x}}=\tau|_{G(\tilde{x})}:G(\tilde{x})\to H(f(\tilde{x}))$ is an isomorphism. So our theorem also extend [1, Theorem 4.3] even if for the finite-dimensional orbifolds.

## References

[1] M. Carlotti, V-manifold maps and the inverse mapping theorem, Rend. Circ. Mat. Palermo (2) 41 (1992) 325–341.
[2] I. Satake, The Gauss–Bonnet Theorem for V-manifolds, J. Math. Soc. Japan 9 (1957) 464–492.
[3] A. Borel, Seminar on Transformation Groups, Princeton University Press, Princeton, 1960.
[4] A. Dress, Newman's theorems of transformation groups, Topology 8 (1969) 203–207.
[5] V. Guillemin, V. Ginzburg, Y. Karshon, Moment Maps, Cobordisms, and Hamiltonian Group Actions, Math. Surveys Monogr., vol. 98, Amer. Math. Soc., Providence, RI, 2002.
[6] K. Fukaya, K. Ono, Arnold conjecture and Gromov–Witten invariants, Topology 35 (1999) 933–1048.